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On the algebraic structure of covariant anomalies and covariant Schwinger terms

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Abstract. The algebraic structure of covariant anomalies and covariant Schwinger terms in an anomalous Yang–Mills theory is investigated. A cohomological characterization is formulated and geometrically interpreted. A new method of determining covariant anomalies and covariant Schwinger terms based on the BRS–anti-BRS complex is presented.

1. Introduction

The phenomenon of anomalies in a quantum field theory has been extensively studied during the last decade. Anomalies in the divergence of the fermionic current and Schwinger terms in the equal time commutator of the gauge group generators were found to be different manifestations of these anomalies [1, 2]. It has been recognized for a long time that anomalies can occur in two different forms, known as consistent and covariant anomalies. Besides perturbative calculations, the former have been investigated by algebraic [3–5] and topological means [6]. Consistent anomalies as well as consistent Schwinger terms are characterized by an algebraic condition, which is of cohomological nature.

On the other hand, covariant anomalies and covariant Schwinger terms have been studied from various viewpoints in the last few years. Bardeen and Zumino [7] have constructed the covariant form of the anomaly in any even-dimensional spacetime. Furthermore the covariant Schwinger term occuring in the equal time commutator of the covariant Gauss law operators has been determined by different methods in [8–10, 13].

In [11] it was shown how the covariant anomaly can be understood in terms of presymplectic geometry on the space of gauge potentials. However, it is not clear how the covariant Schwinger term can be interpreted in this context.

Furthermore, it was realized that both covariant anomalies and covariant Schwinger terms can be derived by enlarging the usual BRS algebra to include an antighost and an anti-BRS operator [12, 13]. So called homotopy operators were introduced which lead to descent equations for the covariant anomaly and the covariant Schwinger term. As a consequence, these terms satisfy a certain algebraic condition which can be viewed as the covariant counterpart of the consistency condition for consistent anomalies. Another approach for determining the covariant descent equations has been proposed in [9] by gauging the (anti-)BRS symmetry.

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For the sake of completeness we should mention that a different characterization of the covariant anomaly has already been suggested in [14].

The aim of this paper is to investigate the algebraic structure of covariant anomalies. In section 2 we shall formulate a new cohomological characterization of the integrated covariant anomaly and covariant Schwinger term. This analysis is based on the results which we have previously obtained in [10]. In section 3 we give a rigorous mathematical treatment of the BRS–anti-BRS approach of [12, 13] in terms of graded differential algebras. This will be a generalization of the algebraic construction of consistent anomalies described in [3]. Thereby the cohomological structure of the covariance condition proposed in [12, 13] is elucidated. Furthermore, we find a new algebraic method of deriving the covariant descent equations. Our method does not rely upon the use of homotopy operators and leads to strong covariance conditions for the covariant anomaly and the covariant Schwinger term. Finally we comment on the relation between these two cohomological characterizations. The relevant definitions are summarized in the appendix.

2. The notion of covariance for anomalies and Schwinger terms

The present section is devoted to showing that both the covariant anomaly and the covariant Schwinger term satisfy an algebraic condition that can be cohomologically interpreted. To do this we shall use the geometric framework which was introduced in [10].

Let $\mathcal{A}$ be the space of connections of a trivial G-bundle P over n-dimensional spacetime M and consider the principal $\mathcal{G}$-bundle $\mathcal{A}(\mathcal{M}, \pi_{\mathcal{A}}, \mathcal{G})$ of all gauge potentials over the gauge orbit space $\mathcal{M}$, where $\mathcal{G}$ is the gauge group with Lie algebra $\operatorname{Lie}\mathcal{G}$.

The consistent anomaly arises from an anomalous continuity equation

$$i_{Y_\xi}\langle J\rangle\ (A) = \operatorname{Anom}(A,\xi) = \int_M \mathrm{d}x\ \operatorname{Anom}^a(A,x)\xi^a(x) \qquad A\in\mathcal{A},\ \xi\in\operatorname{Lie}\mathcal{G} \tag{2.1}$$

for the vacuum expectation value of the consistent fermionic current J in the gauge field background. The integrated consistent anomaly is a functional which is linear in ξ and local in A. $\langle J\rangle$ is considered as a closed one-form over $\mathcal{A}$ and is connected with the generating functional Z by $\langle J\rangle = \mathrm{id}_{\mathcal{A}}Z/Z$. The fundamental vector field generated by an element ξ of the gauge algebra $\operatorname{Lie}\mathcal{G}$ is denoted by Y_ξ, $i_.$ is the substitution operator, and $\mathrm{d}_{\mathcal{A}}$ is the exterior derivative on $\mathcal{A}$. We note that in local coordinates the fundamental vector field reads $Y_\xi = \int \mathrm{d}x\,(D_A)^{ab}_\nu\xi^b(x)(\delta/\delta A^a_\nu(x))$, where D_A denotes the covariant derivative and the indices a, b refer to a basis in the Lie algebra of G. Thus (2.1) can be written in the more familiar form

$$(D_A)^{ab}_\nu\langle J\rangle^\nu_b(x) = -\operatorname{Anom}^a(A,x). \tag{2.2}$$

Finally the Ward operator can be identified with the Lie derivative L_{Y_ξ} with respect to Y_ξ. So (2.1) leads to the Wess–Zumino (WZ) consistency condition [15] for the consistent anomaly

$$L_{Y_\xi}\operatorname{Anom}(.,\eta) - L_{Y_\eta}\operatorname{Anom}(.,\xi) - \operatorname{Anom}(.,[\xi,\eta]) = 0. \tag{2.3}$$

The covariant anomaly $\widetilde{\operatorname{Anom}}$ arises from an analogous equation for the covariant current $\langle\tilde{J}\rangle$

$$i_{Y_\xi}\langle\tilde{J}\rangle\ (A) = \widetilde{\operatorname{Anom}}(A,\xi) = \int_M \mathrm{d}x\,\widetilde{\operatorname{Anom}}^a(A,x)\xi^a(x). \tag{2.4}$$

However, both forms of the anomaly are related to each other by the Bardeen–Zumino polynomial Λ [7], which will here be considered as a one-form on $\mathcal{A}$. This relation is given by

$$\widetilde{\mathrm{Anom}}(A,\xi) = \mathrm{Anom}(A,\xi) + i_{Y_\xi}\Lambda \tag{2.5}$$

where the condition $L_{Y_\xi}\Lambda = -\mathrm{d}_{\mathcal{A}}\,\mathrm{Anom}(\xi)$ is imposed on Λ. In [10] we have interpreted Λ as a connection one-form on a certain line bundle over $\mathcal{A}$ with curvature $\mathcal{F} = \mathrm{d}_{\mathcal{A}}\Lambda$.

In [10] we have calculated the covariant Schwinger term in the commutator of the covariant Gauss law operator in any even dimension of spacetime. Thereby we derived the following formula

$$\tilde{S}_{ab}(x,y) = -\mathrm{i}\frac{\delta}{\delta A_0^b(y)}\widetilde{\mathrm{Anom}}_a(x). \tag{2.6}$$

If we define the vector field $X_\xi = \int \mathrm{d}x\,\xi^a(x)(\delta/\delta A_0^a(x)) \in \mathcal{X}(\mathcal{A})$ and use the relation $i_{Y_\xi}\mathcal{F} = -\mathrm{d}_{\mathcal{A}}\widetilde{\mathrm{Anom}}(\xi)$, the covariant Schwinger term can now be written in a manifestly antisymmetric form

$$\tilde{S}(\xi_1,\xi_2) = \tfrac{1}{2}\mathrm{i}(i_{X_{\xi_2}}i_{Y_{\xi_1}}\mathcal{F} - i_{X_{\xi_1}}i_{Y_{\xi_2}}\mathcal{F}). \tag{2.7}$$

Consistent anomalies and Schwinger terms can be studied within the space $C^q(\mathrm{Lie}\,\mathcal{G}, \mathcal{C}(\mathcal{A}))$ of alternate q-linear maps on $\mathrm{Lie}\,\mathcal{G}$ with values in $\mathcal{C}(\mathcal{A})$, the space of functions on $\mathcal{A}$. (To be precise, one has to restrict to the subset of local functionals [5] on $\mathcal{A}$.) In order to formulate an algebraic condition for covariant terms, we shall view elements of $C^q(\mathrm{Lie}\,\mathcal{G}, \mathcal{C}(\mathcal{A}))$ as maps $\mathcal{A} \to \bigwedge^q(\mathrm{Lie}\,\mathcal{G})^*$ and denote the space of all such maps by $\mathcal{C}(\mathcal{A}, \bigwedge^q(\mathrm{Lie}\,\mathcal{G})^*)$. Here $\bigwedge^q(\mathrm{Lie}\,\mathcal{G})^*$ denotes the qth exterior tensor product of the dual of $\mathrm{Lie}\,\mathcal{G}$.

There is a free right action $\mathcal{R}$ of $\mathcal{G}$ on the product $\mathcal{A} \times \bigwedge^q(\mathrm{Lie}\,\mathcal{G})^*$, given by

$$\mathcal{R}_h(A,\phi) := (A^h, \mathrm{Ad}^*(h^{-1})\mu) \qquad \mu \in \wedge^q(\mathrm{Lie}\,\mathcal{G})^*,\ h \in \mathcal{G} \tag{2.8}$$

with the coadjoint action Ad^* of $\mathcal{G}$ on $\bigwedge^q(\mathrm{Lie}\,\mathcal{G})^*$

$$(\mathrm{Ad}^*(g)\mu)(\xi_1,\ldots,\xi_q) := \mu(\mathrm{Ad}(g^{-1})\xi_1,\ldots,\mathrm{Ad}(g^{-1})\xi_q)$$
$$g \in \mathcal{G},\ \xi_1,\ldots,\xi_q \in \mathrm{Lie}\,\mathcal{G}. \tag{2.9}$$

The corresponding derived representation, denoted by ad^*, is given by

$$(\mathrm{ad}^*(\eta)\mu)(\xi_1,\ldots,\xi_q) = -\sum_{i=1}^{q}\mu(\xi_1,\ldots,[\eta,\xi_i],\ldots,\xi_q) \qquad \eta \in \mathrm{Lie}\,\mathcal{G}. \tag{2.10}$$

An element $\phi \in \mathcal{C}(\mathcal{A}, \bigwedge^q(\mathrm{Lie}\,\mathcal{G})^*)$ is called equivariant if

$$\phi(A^g) = \mathrm{Ad}^*(g^{-1})\phi(A) \tag{2.11}$$

and the space of equivariant maps is denoted by $\mathcal{C}_{\mathrm{eq}}(\mathcal{A}, \bigwedge^q(\mathrm{Lie}\,\mathcal{G})^*)$.

Using (2.3), (2.5) and (2.10) it can easily be verified that

$$(L_{Y_\xi}\widetilde{\mathrm{Anom}})(\eta) = \widetilde{\mathrm{Anom}}([\xi,\eta]) = -(\mathrm{ad}^*(\xi)\widetilde{\mathrm{Anom}})(\eta). \tag{2.12}$$

On the other hand, we find for the covariant Schwinger term (2.7), using the gauge invariance of $\mathcal{F}$, namely $L_{Y_\xi}\mathcal{F} = 0$, and the commutator

$$[X_\xi, Y_\eta] = X_{[\xi,\eta]} \qquad \xi,\eta \in \mathrm{Lie}\,\mathcal{G} \tag{2.13}$$

that

$$(L_{Y_\eta}\tilde{S})(\xi_1,\xi_2) = \tilde{S}([\eta,\xi_1],\xi_2) + \tilde{S}(\xi_1,[\eta,\xi_2]) = -(\mathrm{ad}^*(\eta)\tilde{S})(\xi_1,\xi_2). \tag{2.14}$$

In summary we have proven:

Proposition 1.

(i) $\widetilde{\mathrm{Anom}} \in \mathcal{C}_{\mathrm{eq}}(\mathcal{A}, (\mathrm{Lie}\,\mathcal{G})^*)$

(ii) $\tilde{S} \in \mathcal{C}_{\mathrm{eq}}(\mathcal{A}, \bigwedge^2(\mathrm{Lie}\,\mathcal{G})^*)$.

Let us construct the exterior coadjoint bundle $\mathcal{E}_q^*$ over the gauge orbit space by the commutative diagram

$$\begin{array}{ccc} E_q^* := \mathcal{A} \times \bigwedge^q(\mathrm{Lie}\,\mathcal{G})^* & \longrightarrow & \mathcal{E}_q^* := \mathcal{A} \times_{\mathcal{G}} \bigwedge^q(\mathrm{Lie}\,\mathcal{G})^* \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & \xrightarrow{\pi_{\mathcal{A}}} & \mathcal{M}. \end{array} \tag{2.15}$$

Proposition 1 tells us that $\widetilde{\mathrm{Anom}}$ and $\tilde{S}$ descend to sections of the two vector bundles $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$, respectively. The space of sections $\Gamma(E_q^*)$ of the trivial bundle E_q^* becomes a $\mathcal{G}$-module with respect to the left action

$$(g \cdot \phi)(A) := \mathrm{Ad}^*(g)\phi(A^g) \qquad g \in \mathcal{G},\ \phi \in \mathcal{C}(\mathcal{A}, \wedge^q(\mathrm{Lie}\,\mathcal{G})^*) \tag{2.16}$$

which in its infinitesimal form reads

$$(\theta(\xi)\phi)(A) = (L_{Y_\xi}\phi)(A) + \mathrm{ad}^*(\xi)\phi(A) \qquad \xi \in \mathrm{Lie}\,\mathcal{G}. \tag{2.17}$$

Now we consider the double complex $K^{p,q} := K^p(\mathrm{Lie}\,\mathcal{G}, \Gamma(E_q^*))$ of n co-chains with values in the space of sections of E_q^* with the two coboundary operators

$$\begin{aligned} &\delta : K^p(\mathrm{Lie}\,\mathcal{G}, \Gamma(E_q^*)) \to K^p(\mathrm{Lie}\,\mathcal{G}, \Gamma(E_{q+1}^*)) \\ &(\delta\Phi)(\xi_1, \ldots, \xi_p)(\eta_1, \ldots, \eta_{q+1}) \\ &\quad := \sum_{i=1}^{q+1} (-1)^{i+1} (L_{Y_{\eta_i}} \Phi(\xi_1, \ldots, \xi_p))(\eta_1, \ldots, \hat{\eta}_i, \ldots, \eta_{q+1}) \\ &\quad\quad + \sum_{1 \leqq i < j \leqq q+1} (-1)^{i+j} \Phi(\xi_1, \ldots, \xi_p)([\eta_i, \eta_j], \eta_1, \ldots, \hat{\eta}_i, \ldots, \hat{\eta}_j, \ldots, \eta_{q+1}) \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\delta_\theta : K^p(\mathrm{Lie}\,\mathcal{G}, \Gamma(E_q^*)) \to K^{p+1}(\mathrm{Lie}\,\mathcal{G}, \Gamma(E_q^*)) \\ &(\delta_\theta\Phi)(\xi_1, \ldots, \xi_{p+1}) \\ &\quad := \sum_{i=1}^{p+1} (-1)^{i+1} \theta(\xi_i)(\Phi(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1})) \\ &\quad\quad + \sum_{1 \leqq i < j \leqq p+1} (-1)^{i+j} \Phi([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1}). \end{aligned}$$

Here a hat denotes omission of the corresponding element. Note that the operator δ is just the usual coboundary operator for the consistent case [3–5]. The cohomology groups with respect to δ_θ and δ are denoted by $H_{\delta_\theta}(K^{p,q})$ and $H_\delta(K^{p,q})$, respectively. As it is well known the consistent anomaly belongs to $H_\delta(K^{0,1})$ and finally the consistent Schwinger term gives a class in $H_\delta(K^{0,2})$.

Since $H_{\delta_\theta}(K^{0,q}) \cong \Gamma(\mathcal{E}_q^*)$, we can formulate the covariance condition for anomalies and Schwinger terms cohomologically by the following:

Proposition 2.

(i) $\qquad \widetilde{\text{Anom}} \in H_{\delta_\theta}(K^{0,1})$

(ii) $\qquad \tilde{S} \in H_{\delta_\theta}(K^{0,2})$.

Before closing this section we will to mention an interesting fact about the equations (2.1) and (2.4). There exists an exact sequence

$$0 \to \mathcal{A} \times \operatorname{Lie}\mathcal{G} \xrightarrow{Y} T\mathcal{A} \xrightarrow{T\pi_{\mathcal{A}}} \pi_{\mathcal{A}}^* T\mathcal{M} \to 0 \tag{2.19}$$

of vector bundles over $\mathcal{A}$. Considering the dual of (2.19) and taking the nth exterior tensor product, one obtains the following exact sequence

$$0 \to \wedge^n \pi_{\mathcal{A}}^* T^*\mathcal{M} \to \wedge^n T^*\mathcal{A} \to E_n^* \to 0 \tag{2.20}$$

of vector bundles over $\mathcal{A}$. Let us denote the complex of $\mathcal{G}$-invariant differential forms on $\mathcal{A}$ by $\Omega_{\mathcal{G}}(\mathcal{A})$. Since (2.20) is also an exact sequence of $\mathcal{G}$ vector bundles, one can easily prove the exactness of the following two sequences of sections

$$\begin{aligned} &0 \to \Gamma(\wedge^n \pi_{\mathcal{A}}^* T^*\mathcal{M}) \to \Omega^n(\mathcal{A}) \xrightarrow{\chi} \Gamma(E_n^*) \to 0 \\ &0 \to \Omega^n(\mathcal{M}) \xrightarrow{\pi_{\mathcal{A}}^*} \Omega_{\mathcal{G}}^n(\mathcal{A}) \to \chi_{\mathcal{G}}\Gamma(\mathcal{E}_n^*) \to 0 \end{aligned} \tag{2.21}$$

where $\chi(\alpha)(\xi_1, \ldots, \xi_n) := i_{Y_{\xi_1}} \cdots i_{Y_{\xi_n}} \alpha$, with $\alpha \in \Omega^n(\mathcal{A})$ and $\xi_i \in \operatorname{Lie}\mathcal{G}$. Finally $\chi_{\mathcal{G}}$ is the restriction of χ to $\Omega_{\mathcal{G}}(\mathcal{A})$.

For $n = 1$, these sequences clearly exhibit the geometric relationship between currents and anomalies, which underlies the anomalous continuity equations (2.1) and (2.4).

3. The BRS–anti-BRS complex

In this section we shall generalize the mathematical treatment of [3] in order also to include anti-BRS transformations. Then a new algebraic approach to determining solutions of the covariance condition which was proposed in [12, 13] will be presented.

Let $\pi: P \to M$ be a principal bundle with principal right action R and structure group G, whose Lie algebra will be denoted by $\mathfrak{g}$. The *gauge group* $\mathcal{G}$ is defined to be the group of all fibre preserving automorphisms $\operatorname{Aut}_0(P)$ of P. Equivalently, the gauge group may be identified with the group $C^\infty_{\text{eq}}(P, G)$ of all smooth maps $\chi: P \to G$, which are equivariant with respect to the adjoint action Ad of G onto itself.

Let $S(g) = \oplus_m S_m(g)$ be the symmetric tensor algebra of g with symmetric tensor product $\vee$ and consider the $S(g)$ valued de Rham complex of P, denoted by $\Omega(P, S(g)) = \Omega(P) \otimes S(g)$. This complex admits natural left actions of G and $\mathcal{G}$

$$\begin{aligned} r(g)\alpha &:= (R_g^* \otimes \mathrm{Ad}(g))\alpha \qquad && g \in G \\ \rho(F)\alpha &:= (F^{-1})^*\alpha && F \in \mathcal{G},\ \alpha \in \Omega(P, S(g)) \end{aligned} \tag{3.1}$$

respectively, where $\mathrm{Ad}(g)(u_1 \vee \cdots \vee u_m) = \mathrm{Ad}(g)u_1 \vee \cdots \vee \mathrm{Ad}(g)u_m$ is the induced adjoint action of G on $S(g)$. Together with the bilinear product

$$(\alpha_1 \otimes f_1) \cdot (\alpha_2 \otimes f_2) := (\alpha_1 \wedge \alpha_2) \otimes (f_1 \vee f_2) \qquad \alpha_i \in \Omega(P),\ f_i \in S(g) \tag{3.2}$$

$\Omega(P, S(g))$ becomes a graded commutative differential algebra (cf the appendix). The subcomplex $\Omega(P, g)$ admits the structure of a graded differential Lie algebra with the bracket $[\,,\,]$ defined by

$$\begin{aligned} &[\alpha_1, \alpha_2](X_1, \ldots, X_{p+q}) \\ &\quad = \frac{1}{p!q!} \sum_{\sigma \in \Sigma_{p+q}} (-1)^\sigma [\alpha_1(X_{\sigma(1)}, \ldots, X_{\sigma(p)}), \alpha_2(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})] \end{aligned} \tag{3.3}$$

where $\alpha_1 \in \Omega^p(P, g)$, $\alpha_2 \in \Omega^q(P, g)$, $X_i \in \mathcal{X}(P)$ and Σ_{p+q} denotes the set of permutations of the first $p + q$ numbers.

The *gauge algebra* $\mathrm{Lie}\,\mathcal{G}$ can be identified with $C^\infty_{\mathrm{eq}}(P, g)$, the Lie algebra of smooth ad equivariant maps $\xi: P \to g$. Equivalently, $\mathrm{Lie}\,\mathcal{G}$ can also be considered as the space of all G-invariant, vertical smooth vector fields $\mathcal{X}^G_{\mathrm{ver}}(P)$ on P.

The derived representations of g and $\mathrm{Lie}\,\mathcal{G}$ on $\Omega(P, S(g))$ are given by (we shall denote them by the same symbols)

$$\begin{aligned} r(u)\alpha &:= L_{Z_u}\alpha + \mathrm{ad}(u)\alpha && u \in g \\ \rho(\xi)\alpha &:= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \rho(\exp t\xi)\alpha = -L_{Z_\xi}\alpha \qquad && \alpha \in \Omega(P, S(g)),\ \xi \in \mathrm{Lie}\,\mathcal{G} \end{aligned} \tag{3.4}$$

respectively, where the Lie derivative $L_.$ is taken along the fundamental vector fields Z_u, Z_ξ generated by $u \in g$ and $\xi \in \mathrm{Lie}\,\mathcal{G}$, respectively.

We begin our generalization by considering the triple graded space

$$\begin{aligned} C_m^{p,q,r} &:= C^p(\mathrm{Lie}\,\mathcal{G}, C^q(\mathrm{Lie}\,\mathcal{G}, \Omega^r(P, S_m(g)))) \\ &= (\Omega^r(P, S_m(g)) \otimes \wedge^q(\mathrm{Lie}\,\mathcal{G})^*) \otimes \wedge^p(\mathrm{Lie}\,\mathcal{G})^* \end{aligned} \tag{3.5}$$

of all $C^q(\mathrm{Lie}\,\mathcal{G}, \Omega^r(P, S_m(g)))$ valued, alternate p-linear maps on $\mathrm{Lie}\,\mathcal{G}$. We define a representation Θ_ρ of $\mathrm{Lie}\,\mathcal{G}$ on $C^q(\mathrm{Lie}\,\mathcal{G}, \Omega^r(P, S_m(g)))$ by

$$(\Theta_\rho(\xi)\phi)(\eta_1, \ldots, \eta_q) := \rho(\xi)(\phi(\eta_1, \ldots, \eta_q)) + (\mathrm{ad}^*(\xi)\phi)(\eta_1, \ldots, \eta_q). \tag{3.6}$$

There exists a bilinear product on $C^{*,*,*} := \oplus_m C_m^{*,*,*}$ which is induced by the product given in (3.2) and exterior multiplication in $\bigwedge(\mathrm{Lie}\,\mathcal{G})^*$, namely

$$\begin{aligned} &(\Phi_1 \star \Phi_2)(\xi_1, \ldots, \xi_{p_1+p_2})(\eta_1, \ldots, \eta_{q_1+q_2}) \\ &\quad = \frac{(-1)^{p_1(r_2+q_2)+q_1 r_2}}{p_1!p_2!q_1!q_2!} \sum_{\sigma \in \Sigma_{p_1+p_2}} \sum_{\bar\sigma \in \Sigma_{q_1+q_2}} (-1)^{\sigma+\bar\sigma} \\ &\qquad \times \Phi_1(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p_1)})(\eta_{\bar\sigma(1)}, \ldots, \eta_{\bar\sigma(q_1)}) \\ &\qquad \cdot \Phi_2(\xi_{\sigma(p_1+1)}, \ldots, \xi_{\sigma(p_1+p_2)})(\eta_{\bar\sigma(q_1+1)}, \ldots, \eta_{\bar\sigma(q_1+q_2)}) \end{aligned} \tag{3.7}$$

where $\Phi_1 \in C_m^{p_1,q_1,r_1}$ and $\Phi_2 \in C_m^{p_2,q_2,r_2}$.

Let us define the following three coboundary operators

$\delta_\rho : C^{p,q,r} \to C^{p,q+1,r}$

$$
\begin{aligned}
&(\delta_\rho \Phi)(\xi_1, \dots, \xi_p)(\eta_1, \dots, \eta_{q+1}) \\
&\quad = \sum_{i=1}^{q+1} (-1)^{i+1} \rho(\eta_i)(\Phi(\xi_1, \dots, \xi_p))(\eta_1, \dots, \hat{\eta}_i, \dots, \eta_q) \\
&\qquad + \sum_{1 \leqq i < j \leqq q+1} \Phi(\xi_1, \dots, \xi_p)([\eta_i, \eta_j], \eta_1, \dots, \hat{\eta}_i, \dots, \hat{\eta}_j, \dots, \eta_{q+1})
\end{aligned}
\tag{3.8}
$$

$\delta_{\Theta_\rho} : C^{p,q,r} \to C^{p+1,q,r}$

$$
\begin{aligned}
&(\delta_{\Theta_\rho} \Phi)(\xi_1, \dots, \xi_{p+1})(\eta_1, \dots, \eta_q) \\
&\quad := \sum_{i=1}^{p+1} (-1)^{i+1} (\Theta_\rho(\xi_i)\Phi(\xi_1, \dots, \xi_i, \dots, \xi_{p+1}))(\eta_1, \dots, \eta_q) \\
&\qquad + \sum_{1 \leqq i < j \leqq p+1} \Phi([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{p+1})(\eta_1, \dots, \eta_q)
\end{aligned}
$$

and $\mathrm{d}: C^{p,q,r} \to C^{p,q,r+1}$ which is induced by the exterior derivative on P. The compatibility of these three operators can easily be checked. If we define the modified coboundary operators (BRS and anti-BRS operator, respectively)

$$
\begin{aligned}
s_\rho &:= (-1)^{r+1} \delta_\rho : C^{p,q,r} \to C^{p,q+1,r} \\
s_{\Theta_\rho} &:= (-1)^{q+r+1} \delta_{\Theta_\rho} : C^{p,q,r} \to C^{p+1,q,r}
\end{aligned}
\tag{3.9}
$$

then $C^{*,*,*}$ admits the structure of a triple complex, i.e. there are the relations

$$
\begin{aligned}
&\mathrm{d}^2 = s_\rho^2 = s_{\theta_\rho}^2 = 0 \\
&\mathrm{d}s_\rho + s_\rho \mathrm{d} = \mathrm{d}s_{\theta_\rho} + s_{\theta_\rho}\mathrm{d} = s_{\theta_\rho} s_\rho + s_\rho s_{\theta_\rho} = 0.
\end{aligned}
\tag{3.10}
$$

Finally the total complex $C^* = \bigoplus_{p,q,r} C^{p,q,r}$ with total derivative $\Delta = \mathrm{d} + s_\rho + s_{\Theta_\rho}$ equipped with the product (3.7) becomes a graded commutative differential algebra.

Now we display the BRS and anti-BRS relations in the sub-complex C_1^*. Since $\Omega(P, g)$ is a differential graded Lie algebra and $\bigwedge(\mathrm{Lie}\,\mathcal{G})^*$ is a graded commutative differential algebra, the complex C_1^* becomes a differential graded Lie algebra, where the bracket $[\,,\,]_C$ between homogenous elements reads

$$
\begin{aligned}
&[\psi_1, \psi_2]_C(\xi_1, \dots, \xi_{p_1+p_2}) \\
&\quad = \frac{1}{p_1! p_2!} \sum_{\sigma \in \Sigma_{p_1+p_2}} (-1)^\sigma [\psi_1(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p_1)}), \psi_2(\xi_{\sigma(p_1+1)}, \dots, \xi_{\sigma(p_1+p_2)})]
\end{aligned}
\tag{3.11}
$$

for $\psi_1 \in C_1^{p_1,q_1,r_1}$, $\psi_2 \in C_1^{p_2,q_2,r_2}$ and $\xi_i \in \mathrm{Lie}\,\mathcal{G}$. Here $[\,,\,]$ is the usual bracket in the BRS complex $C^q(\mathrm{Lie}\,\mathcal{G}, \Omega^n(P, g))$ induced by (3.3) and exterior multiplication in $\bigwedge(\mathrm{Lie}\,\mathcal{G})^*$.

We identify the BRS–anti-BRS multiplet $(A, c, \bar{c})$, with $A \in \mathcal{A} \subset C_1^{0,0,1}$, $c \in C_1^{0,1,0}$, with $c(\xi) = \xi$, and finally $\bar{c} \in C_1^{1,0,0}$, with $\bar{c}(\xi) = \xi$, where $\xi \in \mathrm{Lie}\,\mathcal{G}$. Let $D_A = \mathrm{d} + [A, .]$ denote the covariant exterior derivative on P with respect to the connection A. Since $\rho(\xi)A = -D_A\xi$ and $\rho(\xi)\eta = [\xi, \eta]$, we can prove by a direct computation the following:

Proposition 3. Let $(A, c, \bar{c})$ be the BRS–anti-BRS multiplet. Then the following relations hold

$$\begin{aligned} s_\rho A &= -D_A c & s_{\Theta_\rho} A &= -D_A \bar{c} \\ s_\rho c &= -\tfrac{1}{2}[c, c]_C & s_{\Theta_\rho} \bar{c} &= -\tfrac{1}{2}[\bar{c}, \bar{c}]_C \\ s_\rho \bar{c} &= -[c, \bar{c}]_C & s_{\Theta_\rho} c &= 0. \end{aligned}$$

In the literature c and $\bar{c}$ are called the ghost and the antighost field respectively [16]. Note that the BRS–anti-BRS transformations (proposition 3) do not act symmetrically. However, this property will be important for the formulation of the covariance condition.

Now we want to present a simple method of determining covariant anomalies. Let $\beta = A + c + \bar{c} \in C_1^1$ denote the 'universal connection' and let $\Delta^\beta = \Delta + [\beta, .]_C$ be the associated covariant derivative. The curvature of β is $f = \Delta\beta + \frac{1}{2}[\beta, \beta]_C$. As a consequence of the BRS and anti-BRS relations (proposition 3), an analogue of the well known 'Russian' formula holds, namely

$$f = F_A = \mathrm{d}A + \tfrac{1}{2}[A, A]. \tag{3.12}$$

Let us define the algebraic connection $\beta_0 := A + c$ in the differential graded algebra C_1^*. The curvature of β_0, denoted by f_0, is easily calculated by using the BRS–anti-BRS relations, namely

$$f_0 = F_A - D_A \bar{c}. \tag{3.13}$$

Furthermore, we find for the curvature f_t of the family of connections $\beta_t := t\beta + (1-t)\beta_0$

$$f_t = F_A + (t-1) D_A \bar{c} + \frac{t(t-1)}{2}[\bar{c}, \bar{c}]. \tag{3.14}$$

The dual $S_m^*(g)$ can be identified with the space of symmetric m-linear forms on g by setting $I(u_1, \ldots, u_m) = I(u_1 \vee \cdots \vee u_m)$. We choose the subspace $\mathcal{I}_m(g)$ of Ad invariant forms of $S_m^*(g)$, i.e. forms $I \in S_m^*(g)$ satisfying

$$I(\mathrm{Ad}(g)u_1, \ldots, \mathrm{Ad}(g)u_m) = I(u_1, \ldots, u_m). \tag{3.15}$$

Any $I \in \mathcal{I}_m(g)$ induces a linear map $I\colon C_m^{p,q,r} \to C_0^{p,q,r}$ by

$$\begin{aligned} &I(\Phi)(\xi_1, \ldots, \xi_p)(\eta_1, \ldots, \eta_q)(X_1, \ldots, X_n) \\ &\qquad := I(\Phi((\xi_1, \ldots, \xi_p)(\eta_1, \ldots, \eta_q)(X_1, \ldots, X_n)) \end{aligned} \tag{3.16}$$

and fulfills $I \circ \Delta^\gamma = \Delta \circ I$ for any $\gamma \in C_1^1$ [3]. So one obtains, using the Bianchi identity $\Delta^{\beta_t} f_t = 0$ and the relation $(\mathrm{d}/\mathrm{d}t)\beta_t = \Delta^{\beta_t}\bar{c}$

$$\Delta^{\beta_t}(\bar{c} \star \beta_t \star \cdots \star \beta_t) = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}(\beta_t \star \cdots \star \beta_t). \tag{3.17}$$

Applying $I \in \mathcal{I}_m(g)$ on both sides of (3.17) and integrating, finally gives

$$I(f \star \cdots \star f) - I(f_0 \star \cdots \star f_0) = \Delta Q = m\Delta \int_0^1 \mathrm{d}t\, I(\bar{c} \star f_t \star \cdots \star f_t). \tag{3.18}$$

Let us denote $I(f^{\star m}) := I(f \star \cdots \star f)$ and $I(f_0^{\star m}) := I(f_0 \star \cdots \star f_0)$. Decomposing

$I(f^{\star m})$, $I(f_0^{\star m}) \in C_0^{2m}$ and $Q \in C_0^{2m-1}$ into a sum of elements, homogenous in antighost, ghost and form degree

$$\begin{aligned} I(f^{\star m}) &= \sum_{p+q+r=2m-1} I(f^{\star m})_r^{p,q} \\ I(f_0^{\star m}) &= \sum_{p+q+r=2m-1} I(f_0^{\star m})_r^{p,q} \\ Q &= \sum_{p+q+r=2m-1} Q_r^{p,q} \end{aligned} \tag{3.19}$$

equation (3.18) gives rise to the following generalized system of descent equations

$$(I(f^{\star m}) - I(f_0^{\star m}))_{2m-i-j}^{i,j} = s_{\ominus_\rho} Q_{2m-i-j}^{i-1,j} + s_\rho Q_{2m-i-j}^{i,j-1} + \mathrm{d}Q_{2m-i-j-1}^{i,j}. \tag{3.20}$$

Note that $I(f^{\star m})$, $I(f_0^{\star m})$ and Q are equivariant and horizontal with respect to the principal G action and thus project to well defined differential forms on M.

It is evident from (3.12)–(3.14) that

$$I(f^{\star m})_{2m-i-j}^{i,j} = I(f_0^{\star m})_{2m-i-j}^{i,j} = 0 \qquad Q_{2m-i-j-1}^{i,j} = 0 \qquad \text{for } j \neq 0. \tag{3.21}$$

Using (3.21) we can derive the covariance condition from (3.20), namely

$$s_\rho Q_{2m-i-1}^{i,0} = 0 \tag{3.22}$$

for the non-integrated covariant anomalies with antighost number i. This strong integrability condition for covariant anomalies of arbitrary antighost degree has also been obtained in [9] by using a different method. Our treatment of the BRS–anti-BRS structure clearly exhibits the algebraic structure of this covariance condition.

An explicit calculation gives for $i = 1, 2$

$$\begin{aligned} Q_{2m-2}^{1,0} &= m\ I(\bar{c} \star F_A \star \cdots \star F_A) \\ Q_{2m-3}^{2,0} &= -\frac{m(m-1)}{2}\ Q(\bar{c} \star D_A\bar{c} \star F_A \star \cdots \star F_A) \end{aligned} \tag{3.23}$$

the non-integrated covariant anomaly and covariant Schwinger term, respectively.

Before closing this section we show how the constructed solutions of (3.22) transform under infinitesimal gauge transformations. Since $\mathcal{A}$ is an affine space the Lie derivative of a functional ϕ on $\mathcal{A}$ along the fundamental vector field Y_ξ can be determined by calculating

$$(L_{Y_\xi}\phi)(A) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \phi(A + sD_A\xi) \qquad \xi \in \mathrm{Lie}\,\mathcal{G}. \tag{3.24}$$

In general, $Q_{2m-i-1}^{i,0}$ has the following form

$$Q_{2m-i-1}^{i,0} = \sum_{\substack{2q_1+q_2+1=i \\ \sum_i q_i+1=m}} \frac{m!}{q_1!q_2!q_3!} \int \mathrm{d}t\, I(\bar{c} \star \overbrace{f_t^{2,0,0} \star \cdots \star f_t^{2,0,0}}^{q_1 \text{ times}} \star \underbrace{f_t^{1,0,1} \star \cdots \star f_t^{1,0,1}}_{q_2 \text{ times}} \star \underbrace{f_t^{0,0,2} \star \cdots \star f_t^{0,0,2}}_{q_3 \text{ times}}) \tag{3.25}$$

where $f_t^{p,q,r}$ denotes the corresponding component of f_t with respect to the triple grading of C_1^2. Because of

$$L_{Y_\xi} f_t^{2,0,0} = 0 \qquad L_{Y_\xi} f_t^{1,0,1} = (t-1)[D_A\xi, \bar{c}] \qquad L_{Y_\xi} f_t^{0,0,2} = [F_A, \xi] \tag{3.26}$$

and the identity $[[\xi, \bar{c}]_C, \bar{c}]_C + [\bar{c}, [\xi, \bar{c}]_C]_C = [\xi, [\bar{c}, \bar{c}]_C]_C$, which follows from the graded Jacobi identity in C_1^*, a tedious calculation gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} Q_{2m-i-1}^{i,0}(A + sD_A\xi) = \mathrm{ad}^*(\xi)\ Q_{2m-i-1}^{i,0}(A). \tag{3.27}$$

If we define the integrated forms $\widetilde{\mathrm{Anom}} = \int Q_{2m-2}^{1,0}$ and $\tilde{S} = \int Q_{2m-3}^{2,0}$ then (3.27) provides another proof of proposition 2. So we have explicitly shown that the (integrated) solutions of (3.22) admit the cohomological characterization which was established in a different way in section 2.

Appendix

Here we summarize the relevant definitions concerning differential algebras.

Definition 1. A *graded differential algebra* is a graded vector space $A = \oplus_n A^n$ together with a bilinear product $\cdot: A \times A \to A$ and a linear operator $\mathrm{d}: A \to A$ with $\mathrm{d}^2 = 0$, satisfying

(i) $A^p \cdot A^q \subset A^{p+q}$

(ii) $\mathrm{d}A^p \subset A^{p+1}$

(iii) $\mathrm{d}(a \cdot b) = \mathrm{d}a \cdot b + (-1)^p a \cdot \mathrm{d}b \qquad a \in A^p,\ b \in A.$

Definition 2. A *graded commutative differential algebra* is a graded differential algebra A where the product is associative and satisfies

$$a \cdot b = (-1)^{pq} b \cdot a \qquad a \in A^p,\ b \in A^q.$$

Definition 3. A *differential graded Lie algebra* is a graded differential algebra A where the product satisfies

(i) $a \cdot b = (-1)^{pq+1} b \cdot a \qquad a \in A^p,\ b \in A^q$

(ii) $(-1)^{pr} a \cdot (b \cdot c) + (-1)^{qp} b \cdot (c \cdot a) + (-1)^{rq} c \cdot (a \cdot b) = 0$

$$a \in A^p,\ b \in A^q,\ c \in A^r.$$

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